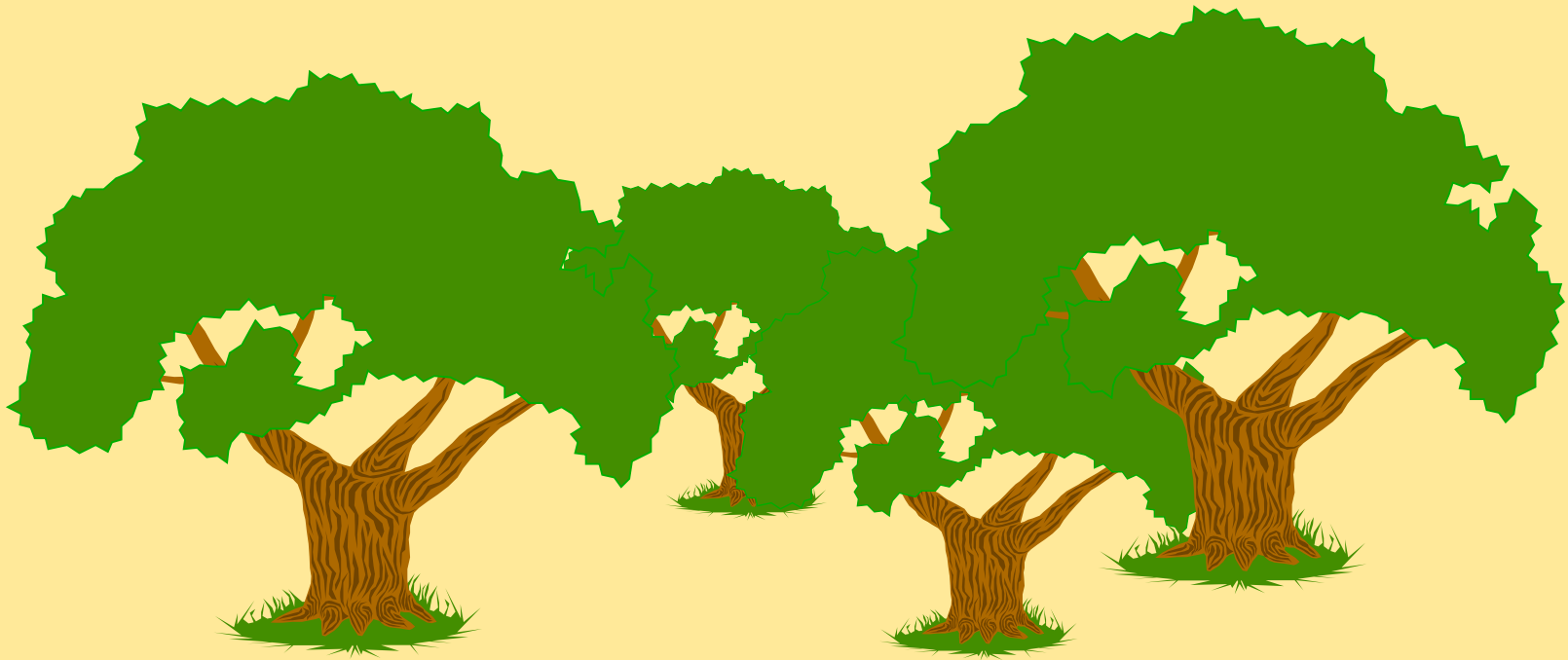


Growth functions



Summary

- ◆ Empirical *versus* biologically based growth functions
- ◆ Theoretical growth functions:
 - ✓ Lundqvist-Korf type
 - ✓ Richards type
 - ✓ Hossfeld IV type
 - ✓ Other growth functions
- ◆ Zeide decomposition of growth functions
- ◆ Formulating growth functions without age explicit
- ◆ Simultaneous modeling of several individuals (trees or stands)

Growth functions

- ◆ The selection of functions - *growth functions* - appropriate to model tree and stand growth is an essential stage in the development of growth models

- ✓ Differential form (growth)

$$\frac{dy}{dt} = f(t)$$

- ✓ Integral form (yield)

$$y = \int f(t)dt$$

Growth functions

- ◆ Growth functions must have a shape that is in accordance with the principles of biological growth:
 - ✓ The curve is limited by yield 0 at the start ($t=0$ ou $t=t_0$) and by a maximum yield at an advanced age (existence of asymptote)
 - ✓ the relative growth rate (variation of the x variable per unit of time and unit of x) presents a maximum at a very early stage, decreasing afterwards; in most cases, the maximum occurs very early so that we can use decreasing functions to model relative growth rate
 - ✓ The slope of the curve increases in the initial stage and decreases after a certain point in time (existence of an inflexion point)

Growth functions

- ◆ Two types of functions have been used to model growth:
 - ✓ *Empirical growth functions*
 - Relationship between the dependent variable - the one we want to model - and the regressors according to some mathematical function - e.g. linear, parabolic, without trying to identify the causes or explaining the phenomenon
 - ✓ *Analitical or functional growth functions*
 - Conceived in terms of the mechanism of the system, usually having an underlying hypothesis associated with the cause or function of the phenomenon described by the response variable
- ◆ The distinction between the two is not sharp and most modeling applications contain both empiricism and mechanism in varying mixtures

Theoretical growth functions

- ◆ Theoretical growth functions have commonly been developed in their growth form - either absolute or relative growth - and the respective yield form has been obtained by integration
- ◆ Generally this approach allows interpretation of the function parameters and helps to impose restrictions on the values that the parameters can take to be biologically consistent
- ◆ Theoretical growth functions are grouped according to their functional form in:
 - ✓ Lundqvist-Korf type
 - ✓ Richards type
 - ✓ Hossfeld IV type
 - ✓ Other growth functions

Theoretical growth functions

Lundqvist-Korf type

Schumacher function

◆ Differential form:

- ✓ The model proposed by Schumacher is based on the hypothesis that the relative growth rate has a linear relationship with the inverse of time² (which means that it decreases nonlinearly with time):

$$\frac{1}{Y} \frac{dY}{dt} = k \frac{1}{t^2} \quad \Leftrightarrow \quad \frac{1}{Y} dY = -k d \frac{1}{t}$$

Schumacher function

◆ Integral form:

$$y = A e^{-k \frac{1}{t}} \quad A = Y_0 e^{k/t_0}$$

- ✓ where the A parameter is the asymptote and (t_0, Y_0) is the initial value
- ✓ the k parameter is inversely related with the growth rate

Lundqvist-Korf function

◆ Differential form:

- ✓ Lundqvist-Korf is a generalization of Schumacher function with the following differential forms:

$$\frac{1}{Y} \frac{dY}{dt} = k \frac{m}{t^{(m+1)}} \quad \Leftrightarrow \quad \frac{1}{Y} dY = -kd \frac{1}{t^m}$$

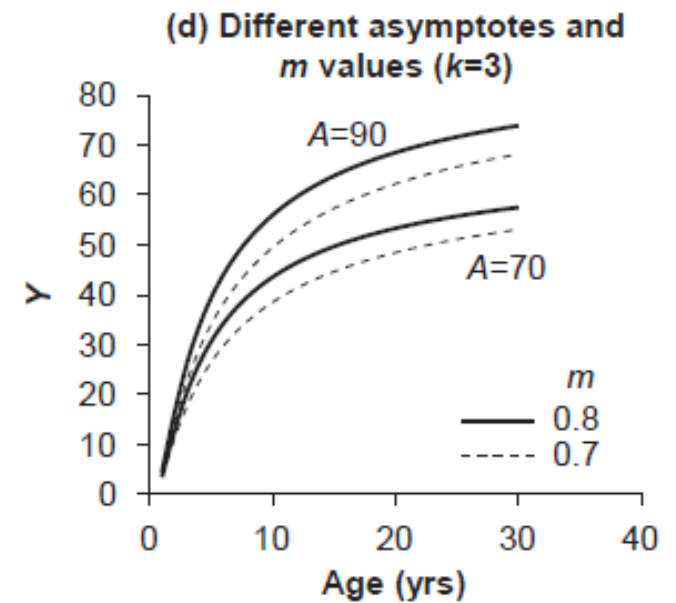
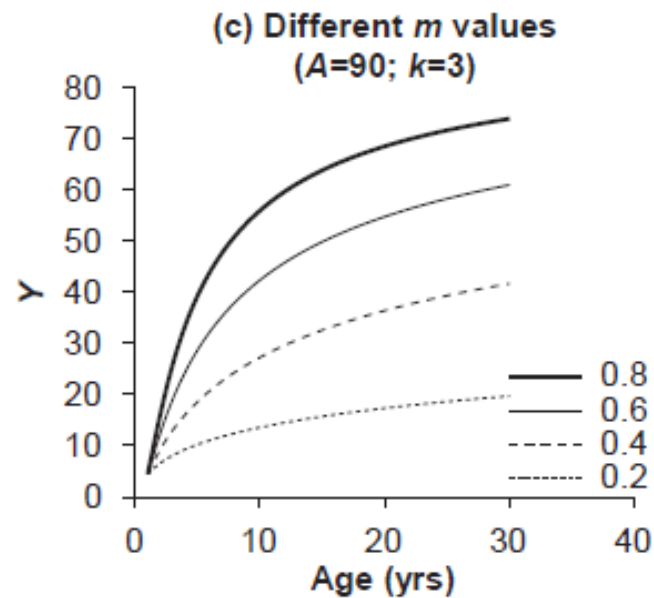
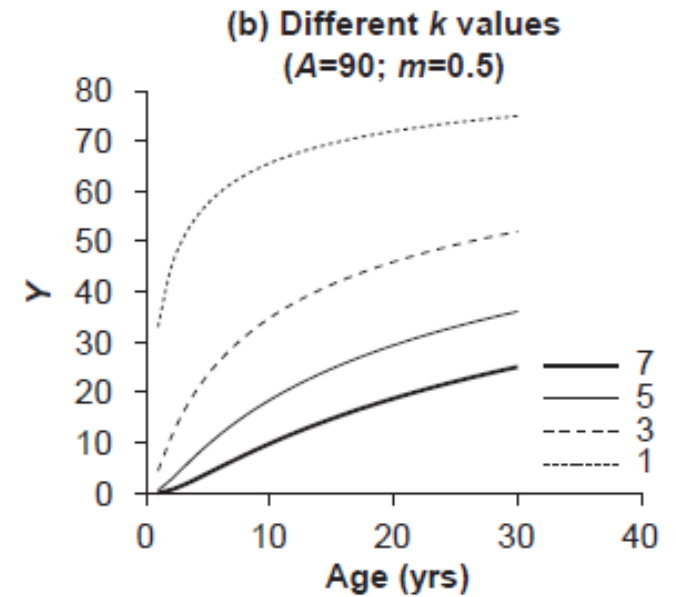
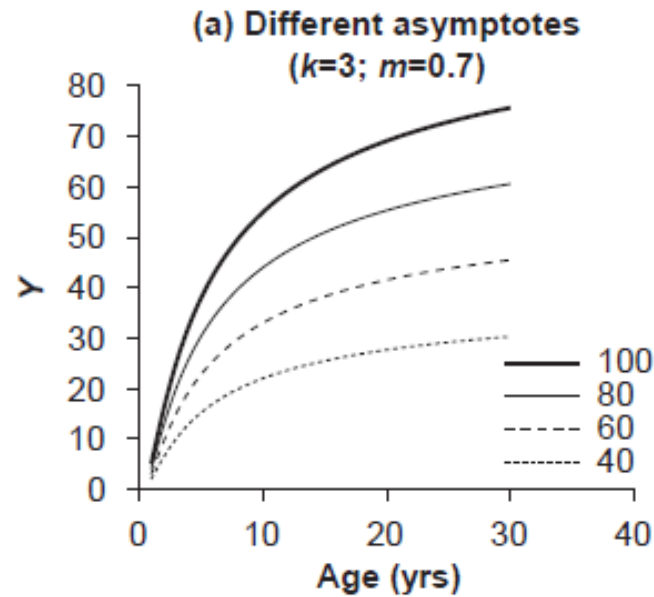
Lundqvist-Korf function

◆ Integral form:

$$Y = A e^{-k \frac{1}{t^m}}$$

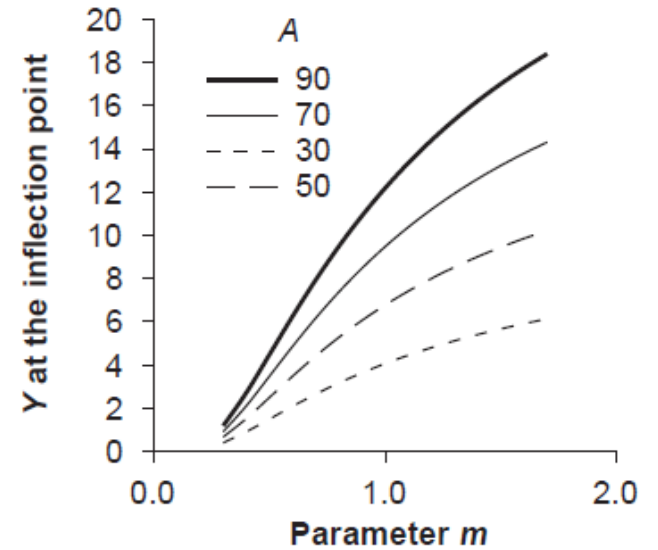
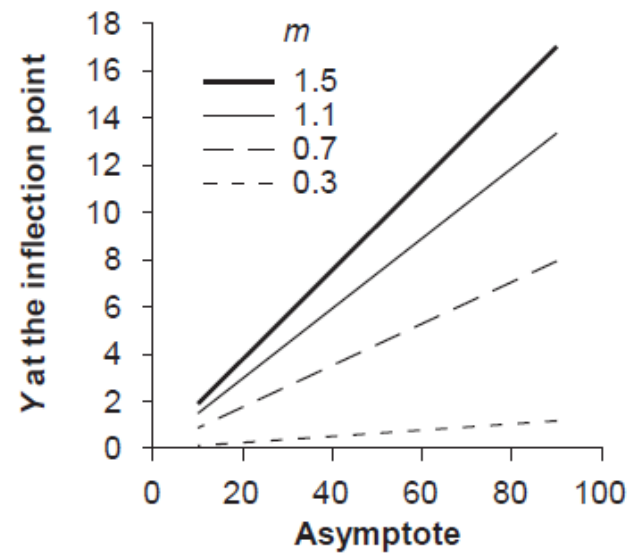
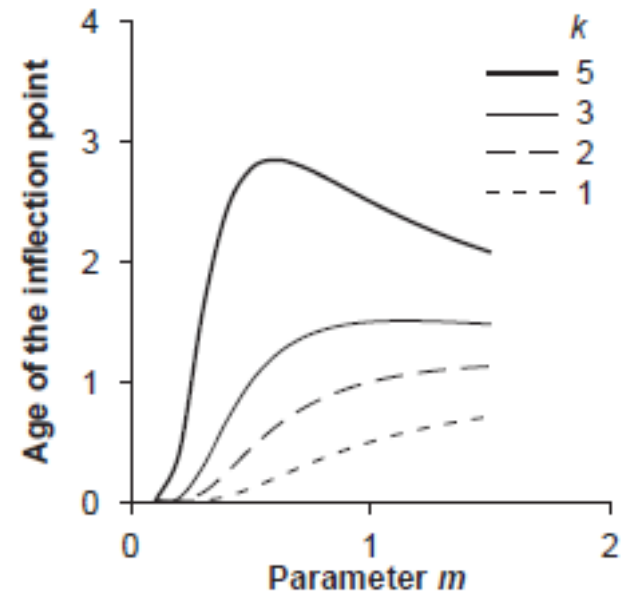
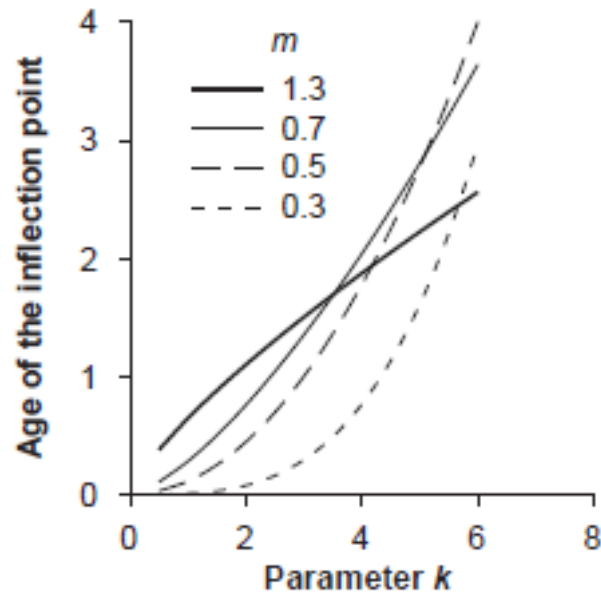
- ✓ The A parameter is the asymptote
- ✓ The k and m parameters are growth rate and shape parameters:
 - k is inversely related with the growth rate
 - m influences the age at which the inflexion point occurs

Lundqvist-Korf function



Lundqvist-Korf function

Location of the inflection point



Relationship between the functions of the Lundqvist-Korf type

◆ Lundqvist function

- ✓ Schumacher's function is a specific case of Lundqvist function for $m=1$

Theoretical growth functions

Richards type

Monomolecular function

◆ Differential form

- ✓ Assumes that the absolute growth rate is proportional to the difference between the maximum yield (asymptote) and the current yield:

$$\frac{dY}{dt} = k(A - Y)$$

Monomolecular function

◆ Integral form:

$$Y = A \left(1 - c e^{-k t} \right)$$

$$c = e^{k t_0} \left(1 - \frac{Y_0}{A} \right)$$

A- asymptote;

k - shape parameter, expressing growth intensity

Logistic function

◆ Differential form:

- ✓ The logistic function is based on the hypothesis that the relative growth rate is the result of the biotic potential k reduced by the current yield or size mY (environmental resistance):

$$\frac{1}{Y} \frac{dY}{dt} = (k - mY)$$

- ✓ Relative growth rate is therefore a decreasing linear function of the current yield

Logistic function

- ◆ Integral form:

$$Y = \frac{A}{(1 + c e^{-kt})}$$

$$c = \frac{Y_0/m}{k - mY_0} e^{kY_0}$$

$$A = \frac{k}{m}$$

- ◆ The inflection point occurs at $t = \log(c)/k$ and $Y = A/2$, which implies that the curve is symmetric around the inflection point

Generalized logistic functions

- ◆ Grosenbaugh (1965):

$$Y = \frac{A}{\left(1 + c e^{-(k_1 t + k_2 t^2 + k_3 t^3)}\right)}$$

- ◆ Monserud (1984)

$$Y = \frac{A}{\left(1 + c e^{-f(X,t)}\right)}$$

✓ where X is a vector of several variables

Gompertz function

◆ Differential form:

- ✓ This function assumes that the relative growth rate is proportional to the difference between the logarithms of the maximum yield and current yield

$$\frac{1}{Y} \frac{dY}{dt} = k(\log A - \log Y)$$

- ✓ Which is equivalent to being inversely proportional to the logarithm of the proportion of current yield to the maximum yield

$$\frac{1}{Y} \frac{dY}{dt} = -k \log\left(\frac{Y}{A}\right)$$

Gompertz function

◆ Integral form:

$$Y = A e^{-c} e^{-kt}$$

$$c = (\log A - \log Y_0) e^{-kt_0} = \log\left(\frac{A}{Y_0}\right) e^{kt_0}$$

Richards function

◆ Differential form

- ✓ The absolute growth rate of biomass (or volume) is modeled as:
 - the *anabolic rate* (construction metabolism), proportional to the photossintethically active area (expressed as an allometric relationship with biomass)
 - the *catabolic rate* (destruction metabolism), proportional to biomass

Anabolic rate _____ $c_1 S = c_1 (c_0 Y^m) = c_2 Y^m$

Catabolic rate _____ $c_3 Y$

Potential growth rate _____ $c_2 Y^m - c_3 Y$

Growth rate _____ $c_4 (c_2 Y^m - c_3 Y)$

S - photossintethically active biomass ; Y - biomass; m - alometric coefficient;
c0,c1,c2,c3 - proportionality coefficients; c4 - efficacy coefficient

Richards function

- ◆ The differential form of the Richards function follows:

$$\frac{dY}{dt} = \eta Y^m - \gamma Y$$

Richards function

◆ Integral form:

- ✓ By integration and using the initial condition $y(t_0)=0$, the integral form of the Richards function is obtained:

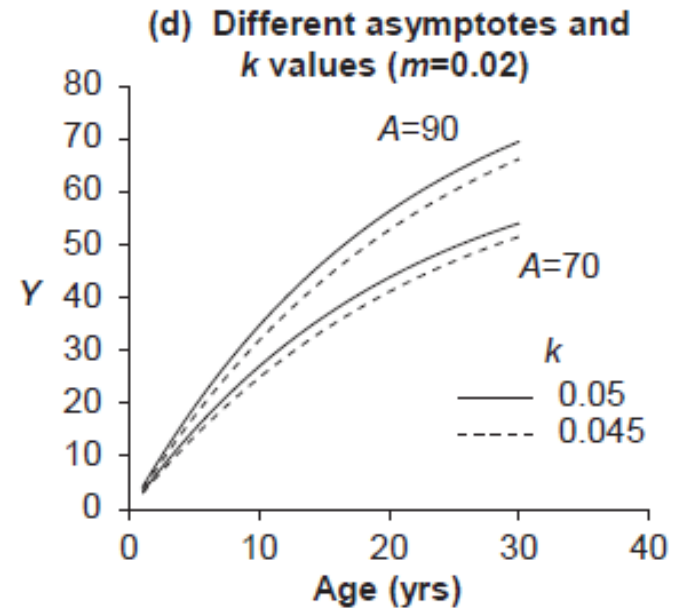
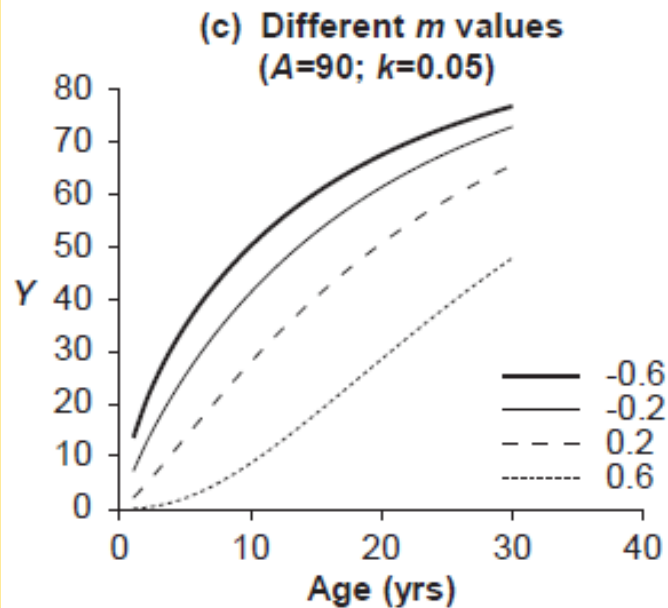
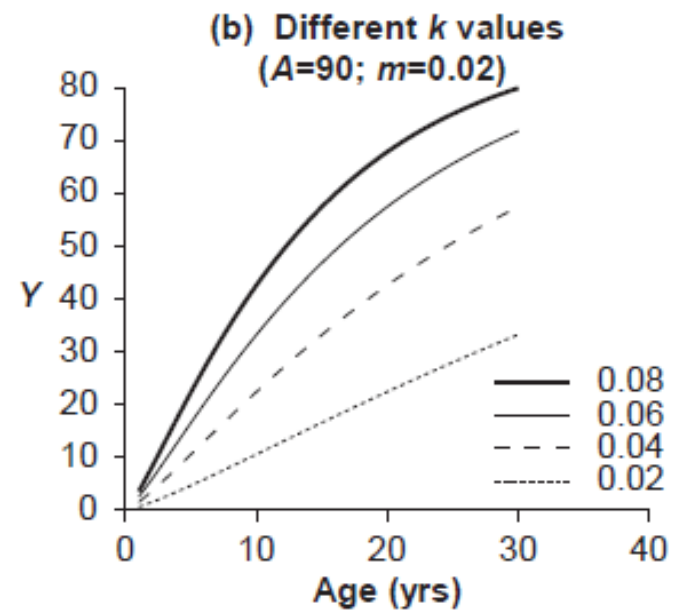
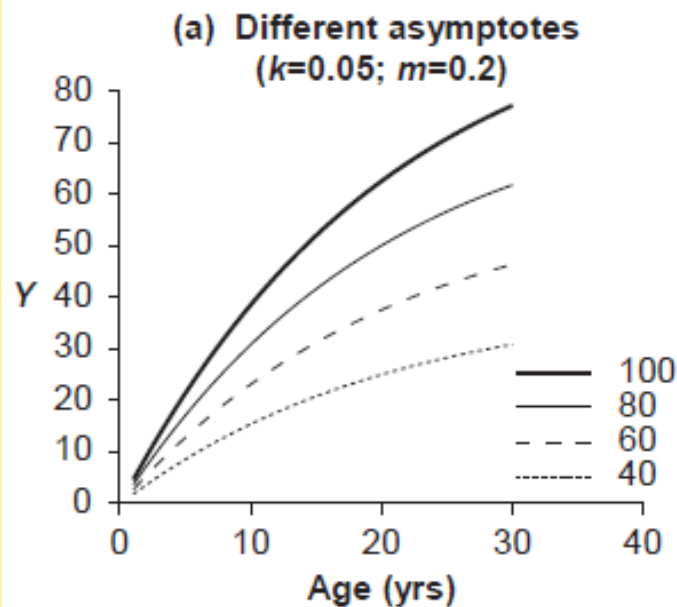
$$Y = A(1 - c e^{-k t})^{\frac{1}{1-m}}$$

with parameters m , c , k and A where: $c = e^{-(1-m)\gamma t_0} = e^{-k t_0}$

$$k = (1-m)\gamma$$

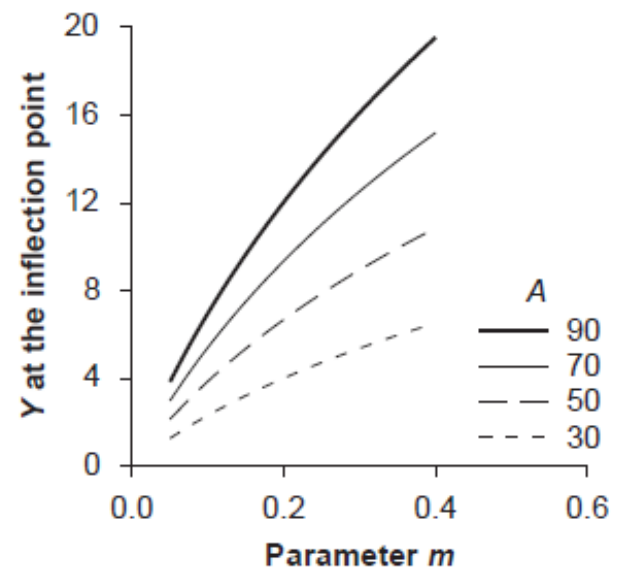
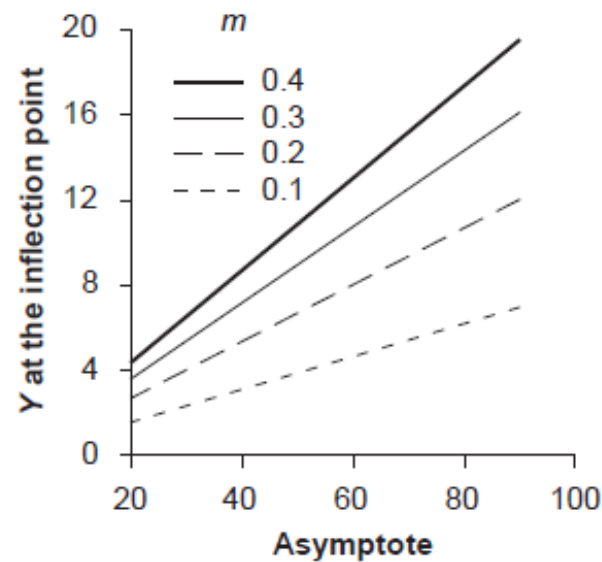
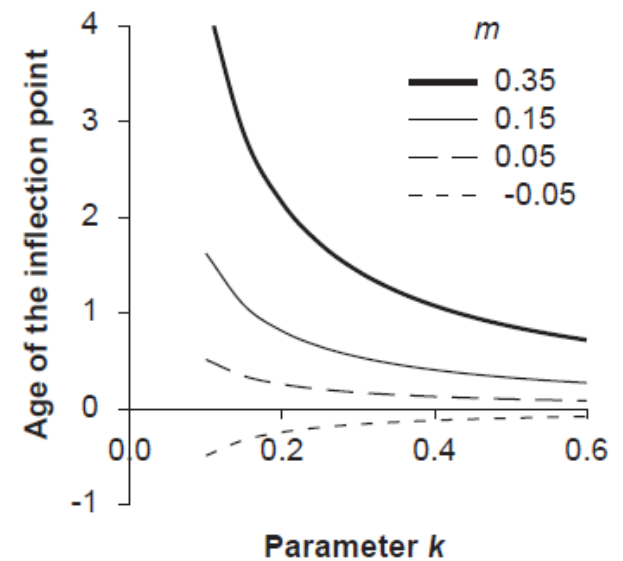
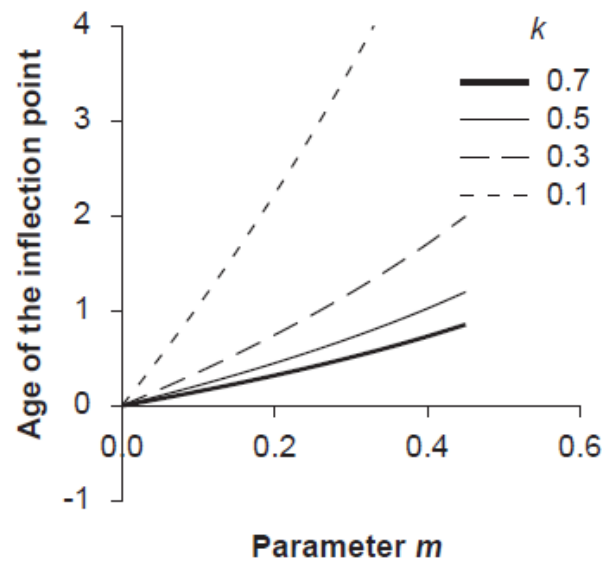
$$A = \left(\frac{\eta}{\gamma}\right)^{\frac{1}{1-m}}$$

Richards function



Richards function

Location of the inflection point



Relationship between the functions of the Richards type

◆ Richards function

- ✓ Monomolecular, logistic and Gompertz are specific cases of Richards function for the m parameter equal to 0, 2, $\rightarrow 1$

Theoretical growth functions

Hossfeld IV type

Hossfeld IV function

- ◆ The Hossfeld IV function is a sigmoid function, originally proposed in 1822 (Zeide 1993), for the description of tree growth:

$$Y = \frac{t^k}{c + t^k/A} = A \frac{t^k}{Ac + t^k}$$

- ◆ The function can also be obtained from the generalized logistic by using $f(X,t) = -k \log(t)$. Consequently some authors designate it as the log-logistic growth function

McDill-Amateis / Hossfeld IV function

◆ Differential form:

- ✓ The variables considered in the development of the growth function and the respective dimensions were:

Variable	dY/dt	t	Y	A
Dimension	L T ⁻¹	T	L	L

where L indicates length, T is time and A is the asymptote

- ✓ Applying differential analysis to these variables, the following differential form is obtained:

$$\frac{dY}{dt} = k \frac{Y}{t} \left(1 - \frac{Y}{A} \right)$$

where k is a parameter related to the growth rate

McDill-Amateis / Hossfeld IV function

◆ Integral form:

$$Y = \frac{A}{1 - \left(1 - \frac{A}{Y_0}\right) \left(\frac{t_0}{t}\right)^k}$$

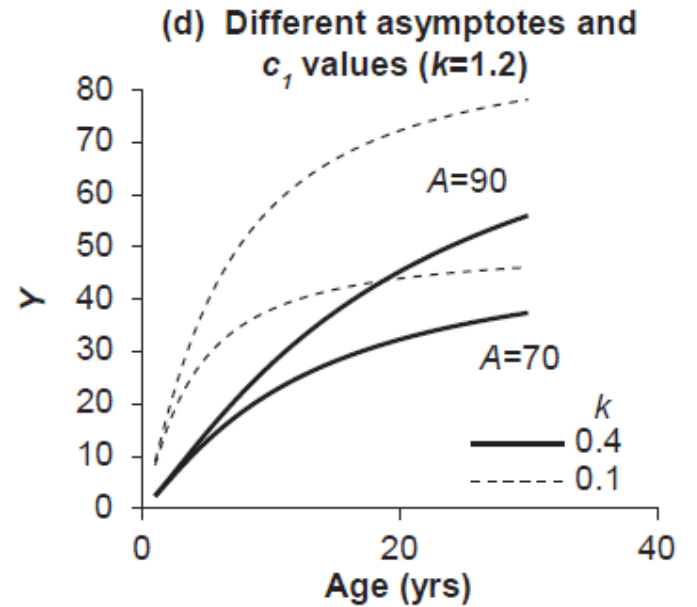
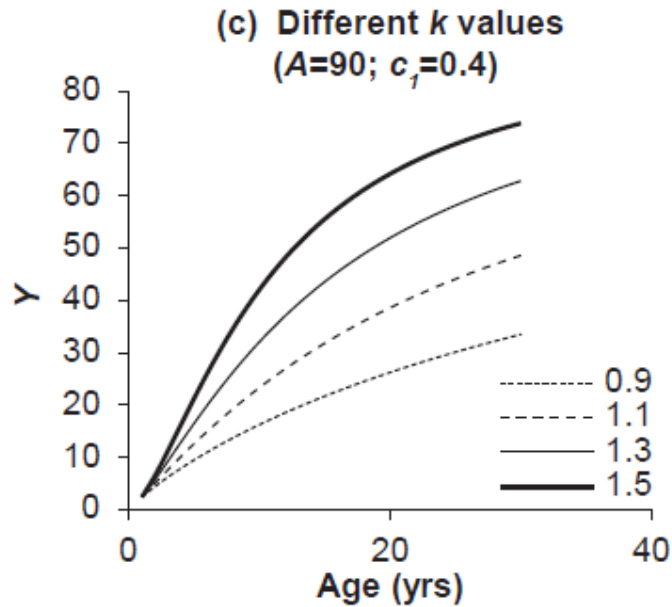
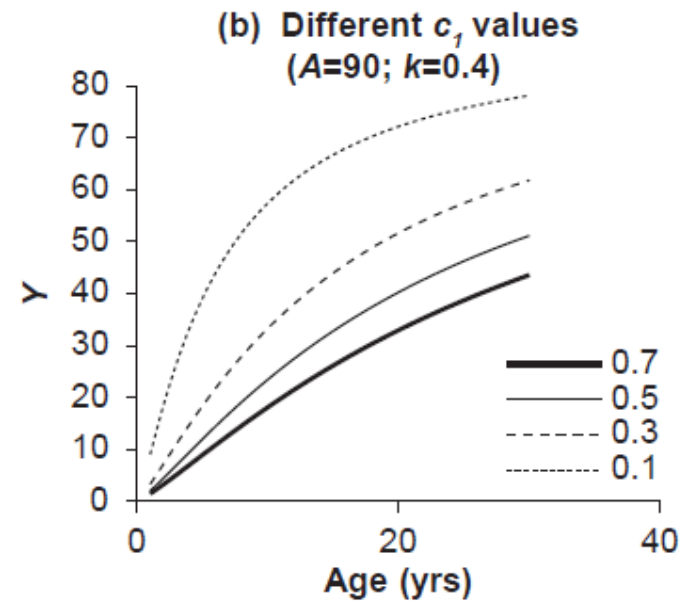
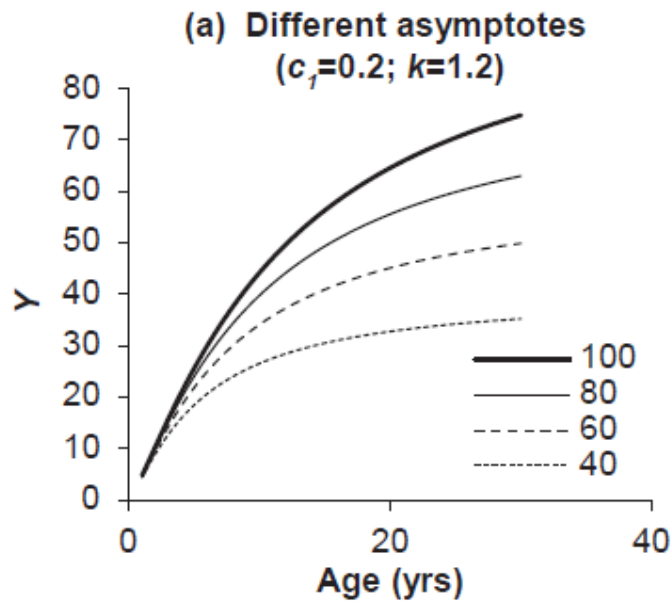
where (t_0, Y_0) is the initial condition and k expresses the growth rate

✓ By making

$$c = \left(\frac{1}{Y_0} - \frac{1}{A}\right) t_0^k$$

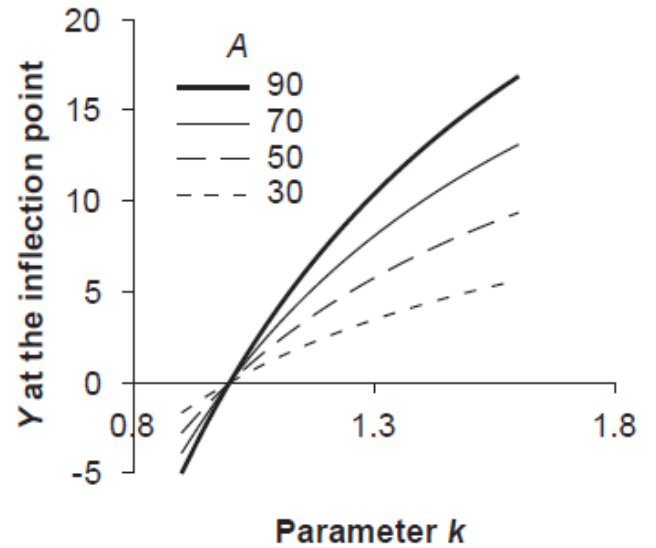
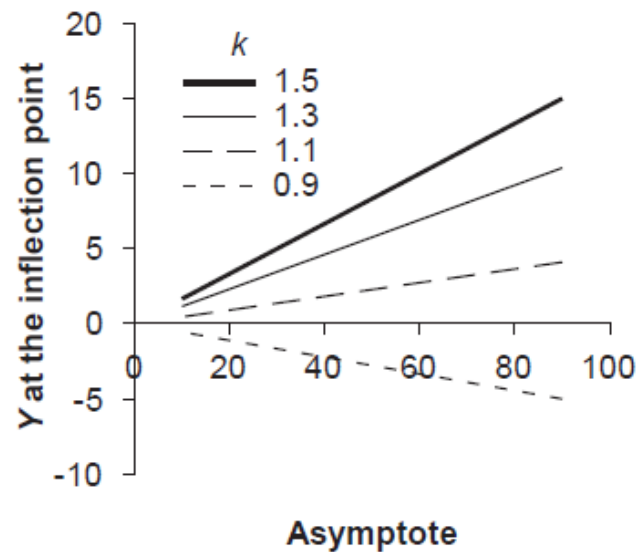
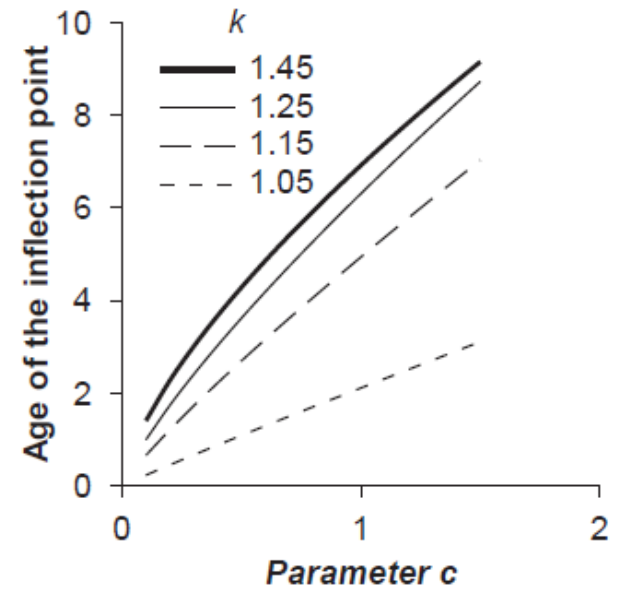
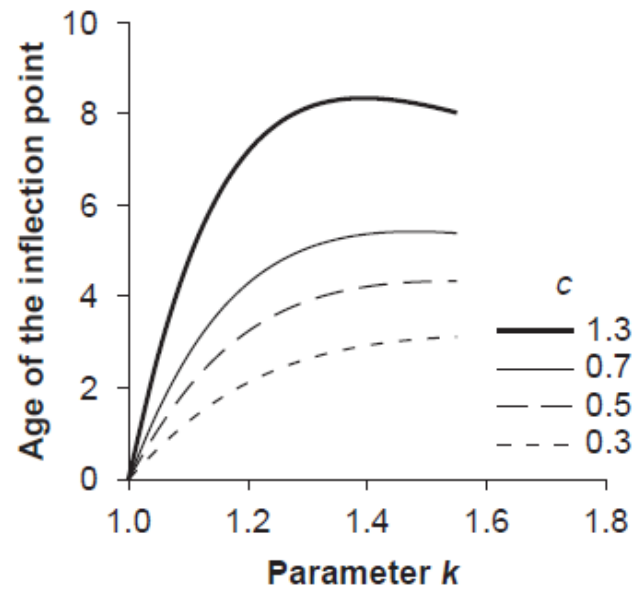
the integral form of the McDill-Amateis function coincides with the Hossfeld IV function

Hossfeld IV function



Hossfeld IV function

Location of the
inflection point



Simultaneous modeling of several individuals (Families of growth functions)

Families of growth functions

- ◆ The fitting of a growth function to data from a permanent plot is straightforward

Example:

- ✓ Fitting the Lundqvist function to basal area and dominant height growth data from a permanent plot

$$Y = A e^{-k \left(\frac{1}{t^n} \right)}$$

A - asymptote

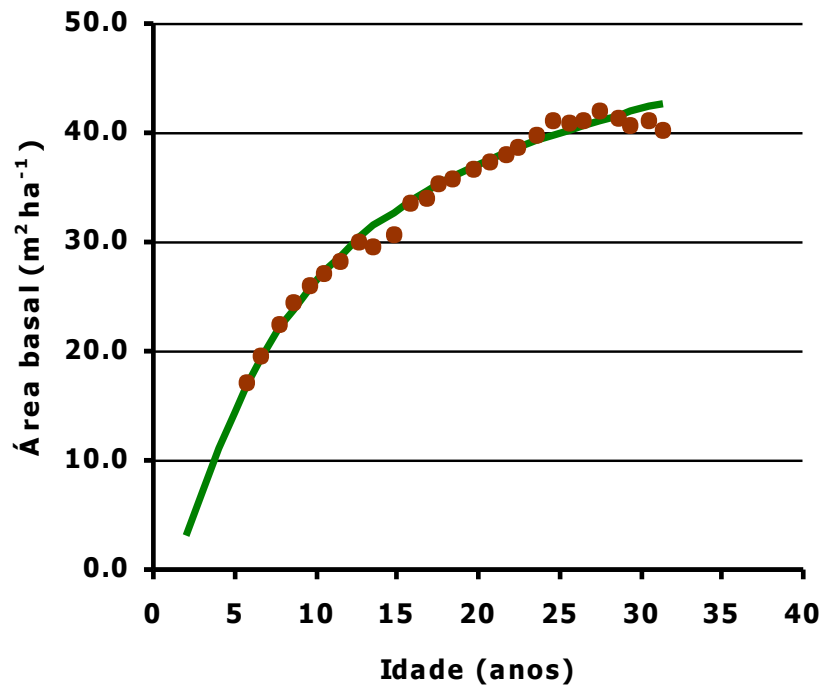
k, n - shape parameters

Growth functions

Basal area

$A = 58.46$, $k = 5.13$, $n = 0.81$

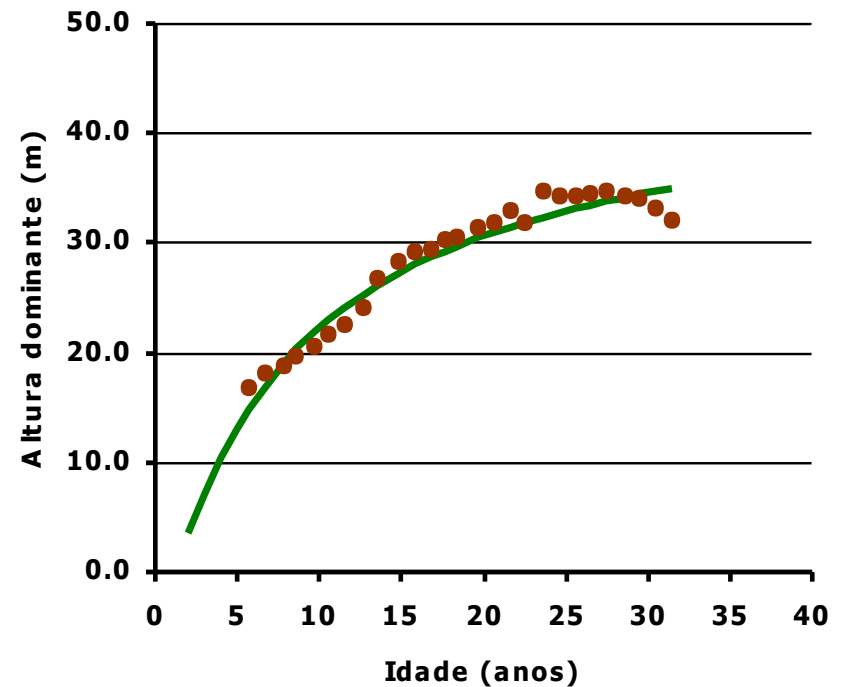
Modelling efficiency = 0.995



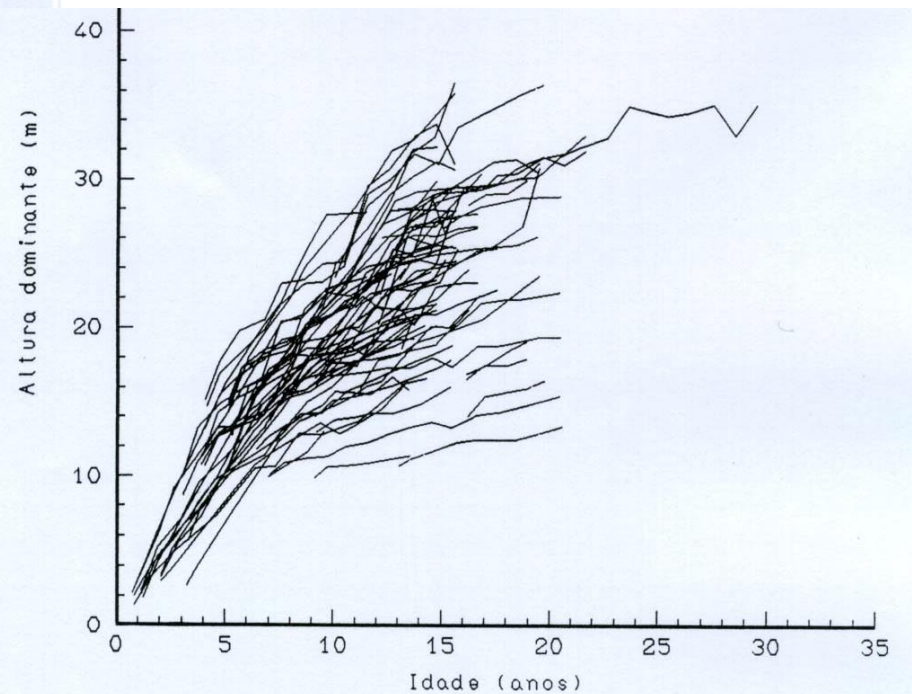
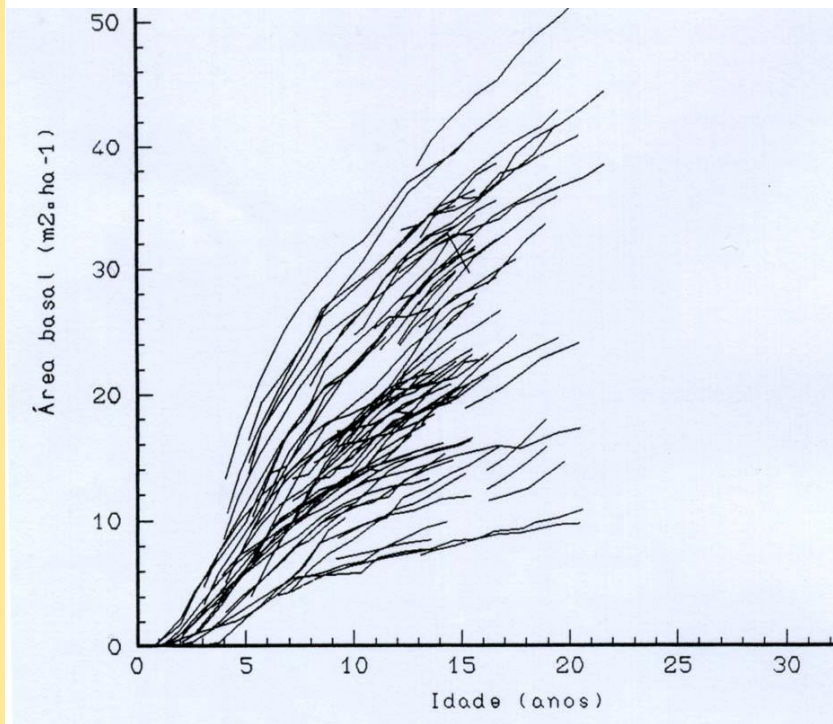
Dominant height

$A = 48.75$, $k = 4.30$, $n = 0.75$

Modelling efficiency = 0.960



But how to model the growth of a series of plots?
This is our objective when developing FG&Y models...



Those plots represent “families” of curves

But how to model the growth of several plots?

- ◆ There are several methods to simultaneously model the growth of several plots:
 - ✓ Using growth functions formulated as difference equations - ADA
 - ✓ Expressing the parameters as a function of site and/or tree/stand variables
 - ✓ Using growth functions formulated as difference equations - GADA
 - ✓ Using mixed models
- ◆ In this course we will just focus the two first methods (information on other methods available at the end of the slides)

Using growth functions formulated as difference equations - ADA

◆ Algebraic difference approach (ADA)

- ✓ When formulating a growth function as a difference equation, it is assumed that the curves belonging to the same “family” differ just by one parameter - the free parameter
- ✓ A growth function with 3 parameters allows for 3 different formulations, usually denoted by the free parameter
- ✓ For example for the Richards function:
 - Richards-A (model with site specific asymptote)
 - Richards-k (model with common asymptote)
 - Richards-m (model with common asymptote)

Using growth functions formulated as difference equations - ADA

Example with the Lundqvist function, formulation with common asymptote and common n parameter, A as free parameter (Kundqvist-A):

$$Y = A e^{-k \frac{1}{t^m}} \Rightarrow A = \frac{Y}{e^{-k \frac{1}{t^m}}} \longrightarrow \frac{Y_2}{e^{-k \frac{1}{t_2^m}}} = \frac{Y_1}{e^{-k \frac{1}{t_1^m}}}$$

$$Y_2 = Y_1 e^{-k \left(\frac{1}{t_2^m} - \frac{1}{t_1^m} \right)} \Leftrightarrow Y = Y_0 e^{-k \left(\frac{1}{t^m} - \frac{1}{t_0^m} \right)}$$

A specific curve of the family is defined by the value of the free parameter

In practice, the free parameter is a function of an initial condition (Y_0, t_0)

Expressing parameters as a function of tree/stand variables

- ◆ Example with the Lundqvist function fit to basal area growth of eucalyptus (GLOBULUS 2.1 model) :

$$G = A e^{-k \left(\frac{1}{t}\right)^m}$$

$$A = A_0 S^2$$

$$k = k_0 + k_1 S + k_2 \frac{N_{pl}}{1000} + k_3 f_e \quad \text{with} \quad f_e = \frac{100}{S \sqrt{N_{pl}}}$$

$$m = m_0 + m_1 \ln(S) + m_2 \frac{N}{1000}$$

Difference equations when the parameters are expressed as a function of tree/stand variables

- ◆ Example with the Lundqvist function fit to basal area growth of eucalyptus with k as free parameter:

$$G = A e^{-k \left(\frac{1}{t}\right)^{m_0 + m_1 N}}$$

- ◆ By using the same method as above we obtain:

$$G_2 = A \left(\frac{G_1}{A}\right)^{\frac{t_1^{m_0 + m_1 N_1}}{t_2^{m_0 + m_1 N_2}}}$$

Formulating growth functions without age explicit

Formulating growth functions without age explicit

- ◆ In many applications age is not known, e.g. in trees that do not exhibit easy to measure growth rings or in uneven aged stands
- ◆ For these cases it is useful to derive formulations of growth functions in which age is not explicit
- ◆ The derivation of these formulations is obtained by expressing t as a function of the variable and the parameters and substituting it in the growth function written for $t+a$ (Tomé et al. 2006)

Formulating growth functions without age explicit

◆ Example with the Lundqvist function

$$Y_t = A e^{-k \frac{1}{t^m}} \quad \longrightarrow \quad t = \left[\frac{-k}{\ln(y_t/A)} \right]^{\frac{1}{m}}$$

$$Y_{t+a} = A e^{-k \frac{1}{(t+a)^m}} \quad \longrightarrow \quad Y_{t+a} = A e^{-k \frac{1}{\left(\left[\frac{-k}{\ln(y_t/A)} \right]^{\frac{1}{m}} + a \right)^m}}$$



The end !!!

Zeide decomposition of growth functions

Zeide decomposition of growth functions

- ◆ Zeide found out that all the growth functions can be decomposed into two components:
 - ✓ Growth expansion - represents the innate tendency towards exponential multiplication and is associated with biotic potential, photosynthetic activity, absorption of nutrients, constructive metabolism, anabolism
 - ✓ Growth decline - represents the constraints imposed by external (competition, limited resources, respiration, and stress) and internal (self-regulatory mechanisms and aging) factors

Zeide decomposition of growth functions

- ◆ The decomposition can be achieved either by a subtraction or a division (subtraction of logarithms) of the two effects
- ◆ All the equations analyzed by Zeide, except Weibull's, are particular cases of the two following forms:
- ◆ LTD $\ln y' = k + p \ln y + q \ln t \leftrightarrow y' = k_1 y^p t^q$
- ◆ TD $\ln y' = k + p \ln y + q t \leftrightarrow y' = k_1 y^p e^{qt}$
where $p > 0$, $q < 0$ and $k = e^k$
- ◆ In both forms the expansion component is proportional to $\ln(y)$ or, in the antilog form, is a power of size
- ◆ In LTD the decline component is proportional to the $\ln$ of age while in TD it is proportional to age

Zeide decomposition of growth functions

- ◆ Zeide proposed a third form in which the declining component is expressed as a function of size instead of age:

$$\ln y' = k + p \ln y + q y \leftrightarrow y' = k_1 y^p e^{qy}$$

- ◆ The three forms are very useful for the direct modeling of tree and/or stand growth - these forms provide some assurance that the resulting model will display appropriate behavior from a biological stand point

Using growth functions formulated as difference equations - GADA

◆ Generalized algebraic difference approach (GADA)

- ✓ One of the problems with ADA is the fact that it originates formulations that differ just by one parameter
- ✓ With GADA it is possible to obtain formulations that have more than one site-specific parameter
- ✓ In GADA parameters are assumed to be function of an unobservable set of variables (denoted by X) that express site differences
- ✓ The equations are then solved by X , which, for a particular site, is substituted in the original equation (X_0)

Using growth functions formulated as difference equations - GADA

◆ Example with the Schumacher function

$$\ln(Y) = \alpha + \frac{\beta}{t}$$

Suppose that $\alpha=X$ and $\beta=\gamma X$, then

$$\ln(Y) = X + \frac{\gamma X}{t} \quad \longrightarrow \quad X = \frac{\ln(Y)}{1+\gamma/t} \quad \longrightarrow \quad X_0 = \frac{\ln(Y_0)}{1+\gamma/t_0}$$

By substituting X_0 into the previous expression, we get

$$\ln(Y) = \ln(Y_0) \frac{t_0(t - \gamma)}{t(t_0 - \gamma)}$$

Using growth functions formulated as difference equations - GADA

◆ Another example with the Schumacher function

Suppose now that $\alpha=X$ and $\beta=X$, then

$$\ln(Y) = X - \frac{\beta}{t} \quad \text{and} \quad \ln(Y) = \alpha - \frac{X}{t} \quad \longrightarrow \quad 2 \ln(Y) = \left(X - \frac{\beta}{t}\right) + \left(\alpha - \frac{X}{t}\right)$$

◆ Solving for X:

$$X = \frac{t[\ln(Y) - \alpha] + \beta}{t - 1} \quad \longrightarrow \quad X_0 = \frac{t_0[\ln(Y_0) - \alpha] + \beta}{t_0 - 1}$$

◆ Finally, substituting X0 in the previous expression

$$\ln(Y) = \alpha - \frac{\beta}{t} + \frac{(t-1)t_0}{(t_0-1)t} \left[\ln(Y_0) - \alpha + \frac{\beta}{t_0} \right]$$

Using mixed-models

- ◆ Mixed-models (linear and non-linear) “split” the model error according to different sources of variation, such as:
 - ✓ Region
 - ✓ Stand
 - ✓ Plots
 - ✓ ...
- ◆ When using a model fitted with mixed-models theory it is possible to calibrate the parameters with random components by measuring a small sample of individuals
- ◆ This means that it is possible to use specific parameters for a particular tree/stand

Which is the best method to model “families” of growth functions?

- ◆ There is no best method to model “families” of growth functions
- ◆ If appropriate the three methods can be combined in order to obtain more flexible growth models